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FIRST STRAIN GRADIENT THEORY OF THERMOELASTICITY

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Abstract—A first strain gradient theory of thermoelasticity is formulated employing a method due to Mindlin. The basic equations for linear dynamical thermoelasticity for infinitesimal motion are obtained and discussed. Wave propagation is considered and an example of a spherical thermal inclusion in an infinite body is solved and the corresponding displacement field and the component of stresses, couple stresses, and double stresses are obtained.

1. INTRODUCTION

The linearized couple-stress theories have become an active field of research in recent years. These theories take into account couple and double stresses which are neglected in classical theory of elasticity. The first study of media with couple-stresses is due to E. and F. Cosserat [1]. The modern derivation of the Cosserat equations has been given by Truesdell and Toupin [2], Toupin [3], and Mindlin and Tiersten [4]. In a subsequent paper Mindlin [5] obtained a more general theory taking into account all the terms of the gradient of the strain tensor in contrast to the previous works where only the gradient of curl of displacement were considered. This generalization had been indicated briefly by Toupin [3]. A derivation of the basic equations based on conservation principles was given recently by Mindlin and Eshel [6]. A general thermodynamical treatment of strain gradient theories was given by Green and Rivlin [7].

In the present work a first strain gradient theory of thermoelasticity for infinitesimal deformation is derived. The derivation follows the method of Mindlin [5] and Mindlin and Eshel [6]. The fundamental equations of linear dynamical thermoelasticity are obtained and discussed. Wave propagation is considered and the dispersion relations of plane waves are calculated. The presentation is concluded with an example of a spherical thermal inclusion in an infinite body.

2. GOVERNING EQUATIONS

In this section the principle of conservation of momentum, angular momentum and energy are employed in the derivation of the equations of the first strain gradient theory of thermoelasticity.

Let t_i and m_i be the components of force and couple, per unit area, acting on the surface S of a body occupying a volume V; and let f_i and C_i be the component of force and couple per unit mass in V, then the principles of balance of linear and angular momentum are expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathring{u}_i \,\mathrm{d}V = \int_S t_i \,\mathrm{d}S + \int_V \rho f_i \,\mathrm{d}V \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho e_{ijk} x_j \mathring{u}_k \,\mathrm{d}V = \int_S (e_{ijk} x_j t_k + m_i)\,\mathrm{d}S + \int_V (e_{ijk} x_j f_k + \rho C_i)\,\mathrm{d}V \tag{2.2}$$

where $\mathbf{u}$ is the displacement vector.

Introducing the stress tensor τ_{ij} and couple stress tensor μ_{ij}, such that

$$t_j = n_i\tau_{ij}, \quad m_j = n_i\mu_{ij}. \tag{2.3}$$

Substitution of (2.3) into (2.1) and (2.2) and applications of divergence theorem lead to the local conservation equations,

$$\tau_{jk,j} + \rho f_k = \rho \mathring{\mathring{u}}_k \tag{2.4}$$

$$\mu_{ij,i} + e_{ijk}\tau_{ki} + \rho C_j = 0. \tag{2.5}$$

We write $\tau_{jk} = \tau_{(jk)} + \tau_{[jk]}$ and evaluate $\tau_{[jk]}$ from (2.5) and substitute it into (2.4), the result is

$$\tau_{(jk),j} - \tfrac{1}{2}\mu^{D}_{il,ij}e_{jkl} + \rho f_k - \tfrac{1}{2}\rho C_{l,j}e_{jkl} = \rho\mathring{\mathring{u}}_k \tag{2.6}$$

where μ^D_{ij} is the deviatoric part of μ_{ij}.

For the helmholtz free energy density, ψ, extending the assumption of Mindlin and Eshel [6], we consider

$$\Psi = \bar{\Psi}(\theta, \epsilon_{ij}, \bar{K}_{ij}, \bar{\bar{K}}_{ijk}) \tag{2.7}$$

$$\theta = T + T_0 \qquad T_0 = \text{temperature of natural state}$$

where

$$T = \text{temperature},$$

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \epsilon_{ji} = \text{Strain} \tag{2.8}$$

$$\bar{K}_{ij} = \tfrac{1}{2}e_{jlk}u_{k,li} = \text{gradient of rotation } (\bar{K}_{ii} = 0) \tag{2.9}$$

$$\bar{\bar{K}}_{ijk} = \tfrac{1}{3}(u_{k,ij} + u_{i,jk} + u_{j,ki}) = \bar{\bar{K}}_{jki} = \bar{\bar{K}}_{kij} = \bar{\bar{K}}_{kji} \tag{2.10}$$

$$= \text{Symmetric part of second gradient of displacement.}$$

Furthermore introducing

$$\eta = -\frac{\partial\bar{\psi}}{\partial\theta} = \text{specific entropy} \tag{2.11}$$

$$\bar{\tau}_{ij} = \rho\frac{\partial\bar{\psi}}{\partial\epsilon_{ij}} = \bar{\tau}_{ji} = \text{stress tensor} \tag{2.12}$$

$$\bar{\mu}_{ij} = \rho\frac{\partial\bar{\psi}}{\partial\bar{K}_{ij}} = \text{deviator of couple stress } (\bar{\mu}_{ii} = 0) \tag{2.13}$$

$$\bar{\bar{\mu}}_{ijk} = \rho\frac{\partial\bar{\psi}}{\partial\bar{\bar{K}}_{ijk}} = \bar{\bar{\mu}}_{jki} = \bar{\bar{\mu}}_{kij} = \bar{\bar{\mu}}_{kji} = \text{double stress tensor,} \tag{2.14}$$

so that

$$\rho\mathring{\psi} = -\rho\eta\mathring{\theta} + \bar{\tau}_{ij}\mathring{\epsilon}_{ij} + \bar{\mu}_{ij}\mathring{\bar{K}}_{ij} + \bar{\bar{\mu}}_{ijk}\mathring{\bar{\bar{K}}}_{ijk}. \tag{2.15}$$

We now adopt the following principle of conservation of energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (\tfrac{1}{2}\rho\mathring{u}_i\mathring{u}_i + \rho e)\,\mathrm{d}V = \int_V (\rho f_i\mathring{u}_i + \tfrac{1}{2}\rho e_{ijk}\mathring{u}_{k,j}C_i + \rho h)\,\mathrm{d}V$$

$$+ \int_s (t_i\mathring{u}_i + \tfrac{1}{2}m_i e_{ijk}\mathring{u}_{k,j} + n_i\bar{\bar{\mu}}_{ijk}\mathring{\epsilon}_{jk} + q_i n_i)\,\mathrm{d}S \qquad (2.16)$$

where

$$e = \text{internal energy density} = \psi - \theta\eta \qquad (2.17)$$

$$q_i = \text{heat flux vector} \qquad (2.18)$$

$$h = \text{heat source distribution.} \qquad (2.19)$$

With (2.3), divergence theorem and chain rule, the surface integral in (2.16) may be converted into a volume integral. Employing (2.6) after some rearrangements we find

$$\int_V \rho\mathring{e}\,\mathrm{d}V = \int_V [(\tau_{(jk)} + \bar{\bar{\mu}}_{ijk,i})\mathring{\epsilon}_{jk} + \mu^D_{ij}\mathring{\bar{K}}_{ij} + \bar{\bar{\mu}}_{ijk}\mathring{\bar{\bar{K}}}_{ijk} + q_{i,i} + \rho h]\,\mathrm{d}V. \qquad (2.20)$$

Finally inserting (2.17) and (2.15) in the left side of (2.20) and equating coefficients of like kinematic variables on both side of the equation, we find

$$\tau_{(jk)} = \bar{\tau}_{jk} - \bar{\bar{\mu}}_{ijk,i} \qquad (2.21)$$

$$\mu^D_{ij} = \bar{\mu}_{ij} \qquad (2.22)$$

$$\rho\theta\mathring{\eta} = q_{i,i} + \rho h. \qquad (2.23)$$

Upon substituting (2.21) and (2.22) into (2.6) we obtain the basic equations of motion of the generalized elastic body

$$\bar{\tau}_{jk,j} - \tfrac{1}{2}e_{jkl}\mu^D_{il,ij} - \bar{\bar{\mu}}_{ijk,ij} + \rho f_k - \tfrac{1}{2}\rho e_{jkl}C_{l,j} = \rho\ddot{u}_k. \qquad (2.24)$$

This coupled with the heat transfer equation (2.23) gives the general equations of the first strain gradient theory of thermoelasticity.

3. LINEAR CONSTITUTIVE EQUATIONS

For a homogeneous centrosymmetric isotropic medium, the most general form of a positive definite helmholtz free energy (2.7) which leads to linear constitutive equations is

$$\rho_0\bar{\psi} = \psi_0 - \rho_0\eta_0 T - \frac{\rho_0\beta}{2T_0}T^2 - \gamma\epsilon_{ii}T + \tfrac{1}{2}\lambda\epsilon_{ii}\epsilon_{jj} + \mu\epsilon_{ij}\epsilon_{ij} + 2\bar{d}_1\bar{K}_{ij}\bar{K}_{ij}$$

$$+ 2\bar{d}_2\bar{K}_{ij}\bar{K}_{ji} + \tfrac{3}{2}\bar{a}_1\bar{\bar{K}}_{iij}\bar{\bar{K}}_{kkj} + \bar{a}_2\bar{\bar{K}}_{ijk}\bar{\bar{K}}_{ijk} + \bar{f}e_{ijk}\bar{K}_{ij}\bar{\bar{K}}_{kll}. \qquad (3.1)$$

Now from (2.11)–(2.14) the specific entropy and constitutive equations may be easily obtained

$$\eta = \eta_0 + \beta T/T_0 + \gamma\epsilon_{ii}/\rho_0 \tag{3.2}$$

$$\bar{\tau}_{pq} = (\lambda\epsilon_{ii} - \gamma T)\delta_{pq} + 2\mu\epsilon_{pq} \tag{3.3}$$

$$\mu^D_{pq} = 4\bar{d}_1\bar{K}_{pq} + 4d_2\bar{K}_{qp} + \bar{f}e_{pqi}\bar{\bar{K}}_{ijj} \tag{3.4}$$

$$\bar{\bar{\mu}}_{pqr} = \bar{a}_1(\bar{\bar{K}}_{iir}\delta_{pq} + \bar{\bar{K}}_{iip}\delta_{qr} + \bar{\bar{K}}_{iiq}\delta_{rp}) + 2\bar{a}_2\bar{\bar{K}}_{pqr} + \tfrac{1}{3}\bar{f}\bar{K}_{ij}(\delta_{pq}e_{ijr} + \delta_{qr}e_{ijp} + \delta_{rp}e_{ijq}). \tag{3.5}$$

When (3.2)–(3.5) together with (2.21) and (2.22) are inserted in (2.6) and ϵ_{ij}, $\bar{K}_{ij}$ and $\bar{\bar{K}}_{ijk}$ are replaced by their expressions in terms of u_i, we find the displacement equation of motion

$$(\lambda + 2\mu)(1 - 1_1^2\nabla^2)\nabla\nabla\cdot\mathbf{u} - \mu(1 - 1_2^2\nabla^2)\nabla\times\nabla\times\mathbf{u} + \rho\mathbf{f} + \tfrac{1}{2}\rho\nabla\times\mathbf{C} - \gamma\nabla T = \rho\ddot{\mathbf{u}} \tag{3.6}$$

where

$$1_1^2 = (3\bar{a}_1 + 2\bar{a}_2)/(\lambda + 2\mu) \tag{3.7}$$

$$1_2^2 = (3\bar{d}_1 + \bar{a}_1 + 2\bar{a}_2 - \bar{f})/3\mu. \tag{3.8}$$

The positive definiteness of the Helmholtz free energy, $\bar{\psi}$ imposes the following restrictions on the coefficients[6, 8],

$$\begin{aligned} &\mu > 0, && 3\lambda + 2\mu > 0, -\bar{d}_1 < \bar{d}_2 < \bar{d}_1 \\ &\bar{a}_2 > 0, && 5\bar{a}_1 + 2\bar{a}_2 > 0, 5\bar{f}^2 < 6(\bar{d}_1 - \bar{d}_2)(5\bar{a}_1 + 2\bar{a}_2) \\ &\beta \geq 0, && \gamma \geq 0 \end{aligned} \tag{3.9}$$

which imply

$$1_1^2 > 0, \quad 1_2^2 > 0. \tag{3.10}$$

Using the expression (3.2) for the entropy in (2.23) gives the heat transfer equation in terms of temperature

$$\rho\theta(\beta\mathring{T}/T_0 + \gamma\mathring{\epsilon}_{ii/\rho_0}) = k\nabla^2 T + \rho h \tag{3.11}$$

where the Fourier law of conduction is employed, i.e.

$$q_i = kT,o.$$

Assuming that $T \ll T_0$ the equation (3.11) may be linearized

$$\rho\beta\mathring{T} + (\gamma T_0)\nabla\cdot\mathring{\mathbf{u}} = k\nabla^2 T + \rho h. \tag{3.12}$$

Note that in the infinitesimal motion theory $\rho \simeq \rho_0$. Equations (3.6) and (3.12) are the basic equations of the first strain gradient theory of thermoelasticity.

Introducing the scalar potential ϕ and the vector potential $\mathbf{A}$ such that

$$\mathbf{u} = \nabla\phi + \nabla\times\mathbf{A}. \tag{3.13}$$

The heat transfer equation (3.12) now becomes

$$\rho\beta\mathring{T} + (\gamma T_0)\nabla^2\mathring{\phi} = k\nabla^2 T + \rho h. \tag{3.14}$$

The equation of motion (3.6) decomposes into two equations in terms of potentials

$$(\lambda + 2\mu)(1 - 1_1{}^2\nabla^2)\nabla^2\phi - \gamma T + \rho f_\phi = \rho\overset{\circ\circ}{\phi} \tag{3.15}$$

$$\mu(1 - 1_2{}^2\nabla^2)\nabla^2\mathbf{A} + \rho\mathbf{f}_A + \tfrac{1}{2}\rho\mathbf{C} = \rho\overset{\circ\circ}{\mathbf{A}} \tag{3.16}$$

where we have assumed

$$f = \nabla f_\phi + \nabla \times \mathbf{f}_A. \tag{3.17}$$

It is interesting to note that the field equation for the vector potential is decoupled from those of the scalar potential and heat transfer.

4. THERMOELASTIC WAVES

The governing equations of a linear, isotropic, homogeneous first gradient thermoelastic solid was obtained in the previous section. In the present section the dispersion of thermoelastic waves is considered. Rewriting the basic equations

$$C_1{}^2(1 - 1_1{}^2\nabla^2)\nabla^2\phi - \gamma_1 T + f_\phi = \ddot{\phi} \tag{4.1}$$

$$C_2{}^2(1 - 1_2{}^2\nabla^2)\nabla^2\mathbf{A} + \mathbf{f}_A + \tfrac{1}{2}\mathbf{C} = \ddot{\mathbf{A}} \tag{4.2}$$

$$\alpha\nabla^2\mathbf{T} - \eta\nabla^2\mathring{\phi} + Q = \mathring{T} \tag{4.3}$$

where

$$C_1{}^2 = \frac{\lambda + 2\mu}{\rho}, \quad C_2{}^2 = \frac{\mu}{\rho} \tag{4.4}$$

are respectively, the velocities of irrotational and equivoluminal isothermal classical elastic waves, and

$$\gamma_1 = \gamma/\rho, \qquad \alpha = k/\rho\beta, \qquad \eta = \gamma T_0/\rho, \qquad Q = h/\beta. \tag{4.5}$$

For vanishing body force, body couple and heat source, combining (4.1) and (4.3) we see that both displacement potential ϕ and temperature satisfy the following sixth order equation.

$$\left\{\left[C_1{}^2(1 - 1_1{}^2\nabla^2)\nabla^2 - \frac{\partial^2}{\partial t^2}\right]\left(\alpha\nabla^2 - \frac{\partial}{\partial t}\right) - \gamma_1\eta\frac{\partial}{\partial t}\nabla^2\right\}\{\phi, T\} = 0. \tag{4.6}$$

The coupling of equivoluminal waves with temperature field will in general occur through the boundary conditions. However the field equation for **A** is uncoupled and, therefore, in the study of wave propagation in an infinite medium there is no need to give any solutions of (4.2). We therefore turn our attention to the coupled equations (4.1) and (4.3) or alternatively (4.6) for ϕ and T.

A plane harmonic wave has the general form

$$\{\phi, T\} = \{\phi^*, T^*\} \exp\{i(\mathbf{k}\cdot x - \omega t)\} \tag{4.7}$$

where $\mathbf{k}$ is the wave vector and ω is the frequency.

Upon substituting (4.7) into (4.6) we obtain the following dispersion relation,

$$[C_1^2(1 + 1_1^2 k^2)k^2 - \omega^2](\alpha k^2 - i\omega) - \gamma_1 \eta k^2 \omega = 0. \tag{4.8}$$

For $1_1 = 0$ the above reduces to the classical dispersion relation of thermoelasticity (see, e.g. Erigen[8]) and for $\eta = 0$ we find the dispersion relation of irrotational waves of the first strain gradient theory of Mindlin[5].

5. SPHERICAL THERMAL INCLUSION IN AN INFINITE BODY

We conclude our presentation by solving an example. Let us consider an infinite body of first gradient elastic media with a spherical thermal inclusion of radius a. The temperature within the sphere is assumed to be a constant T_1 and the rest of the body is kept at zero temperature. We would like to find the displacement field as well as the stress, couple stress, and double stress distributions which are produced due to such a discontinuous temperature distribution.

In the stationary state and in the absence of body force the equations of thermoelasticity (4.1)–(4.3) simply become

$$C_1^2(1 - 1_1^2\nabla^2)\nabla^2\phi - \gamma_1 T = 0. \tag{5.1}$$

In the present example the temperature field is given by

$$T = T_1 H(a - r) \tag{5.2}$$

where H is the Heaviside unit step function.

Due to the symmetry of temperature distribution ϕ is only a function of r. The general solution of (5.1) in this case is easily obtained by the Green function method, i.e.

$$\phi(r) = -\frac{\gamma_1}{1_1 C_1^2 r}\int_0^\infty r_1\,dr_1 G_1(r, r_1)\int_0^\infty r_2^2\,dr_2 G_2(r_1, r_2)T(r_2) \tag{5.3}$$

where

$$G_1(r, r_1) = \begin{cases} e^{-r/1_1}\sinh r_1/1_1 & r > r_1 \\ e^{-r_1/1_1}\sinh r/1_1 & r_1 > r \end{cases} \tag{5.4}$$

$$G_2(r_1, r_2) = \begin{cases} 1/r_1 & r_1 > r_2 \\ 1/r_2 & r_2 > r_1 \end{cases}. \tag{5.5}$$

Upon substituting (5.2) into (5.3) and carrying out the corresponding integration we find,

$$\phi = -\frac{\gamma_1 T_0}{C_1^2}\Big\{\Big[-1/6r^2 - 1_1^2 + \frac{a^2}{2} + \exp\{-a/1_1\}\sinh r/1_1\Big(a\frac{1_1^2}{r} + \frac{1_1^3}{r}\Big)\Big]H(a - r)$$
$$+\Big[\frac{a^3}{3r} + \exp\{-r/1_1\}\Big(-\frac{a\,1_1^2}{r}\cosh a/1_1 + \frac{1_1^3}{r}\sinh a/1_1\Big)\Big]H(r - a)\Big\}. \tag{5.6}$$

The only nonzero component of the displacement field is

$$u_r = \frac{\partial\phi}{\partial r}$$

$$= -\frac{\gamma_1 T_0}{C_1^2}\left\{\left[-\frac{r}{3} + \exp\{-a/1_1\}\left(\frac{1}{1_1 r}\cosh r/1_1 - \frac{1}{r^2}\sinh r/1_1\right)\right]H(a-r)\right.$$

$$\left. + \left[-\frac{a^3}{3r^2} - \exp\{-r/1_1\}(-a\,1_1^2\cosh a/1_1 + 1_1^3\sinh a/1_1)\left(\frac{1}{1_1 r} + \frac{1}{r_2}\right)\right\} H(r-a). \quad (5.7)$$

The corresponding nonzero terms of the strain tensor, gradient of rotation tensor and second gradient of displacement tensor are

$$\epsilon_{11} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{22} = \epsilon_{33} = u_r/r \quad (5.8)$$

$$\bar{\bar{K}}_{111} = \frac{\partial^2 u_r}{\partial r^2}, \ \bar{\bar{K}}_{122} = \bar{\bar{K}}_{212} = \bar{\bar{K}}_{221} = \bar{\bar{K}}_{133} = \bar{\bar{K}}_{313} = \bar{\bar{K}}_{331} = \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \quad (5.9)$$

where subscript 1, 2, 3 corresponds to r, θ, ϕ spherical coordinates system. Note that all the components of gradient of rotation are zero since the rotation itself is identically zero due to the existance of displacement scalar potential.

The corresponding nonzero components of the stress deviatoric part of couple stress and double stress are

$$\bar{\tau}_{11} = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{2u_r}{r}\right) - \gamma T + 2\mu\frac{\partial u_r}{\partial r}, \quad (5.10)$$

$$\bar{\tau}_{22} = \bar{\tau}_{33} = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{2u_r}{r}\right) - \gamma T + 2\mu\frac{u_r}{r},$$

$$\mu^D_{23} = -\mu^D_{32} = \bar{f}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} - \frac{2u_r}{r^2}\right), \quad (5.11)$$

$$\bar{\bar{\mu}}_{111} = 3\bar{a}_1\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} - \frac{2u_r}{r^2}\right) + 2\bar{a}_2\frac{\partial^2 u_r}{\partial r^2}$$

$$\bar{\bar{\mu}}_{122} = \bar{\bar{\mu}}_{212} = \bar{\bar{\mu}}_{221} = \bar{\bar{\mu}}_{133} = \bar{\bar{\mu}}_{313} = \bar{\bar{\mu}}_{331} = \bar{a}_1\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} - \frac{2u_r}{r^2}\right) + 2\bar{a}_2\left(\frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right). \quad (5.12)$$

Direct substitution of (5.7) into (5.10)–(5.12) gives the explicit form of the stresses.

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